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Dipole electromagnetic radiation in a Schwarzschild space: the wavetail to order *m*/*r*

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Abstract. We consider Maxwell's equations in a Schwarzschild space and we determine the retarded solution to order m^2/r^2 of an electric dipole located at the centre of symmetry. The dipole moment $p(t)$ is assumed to be variable in the interval $t_1 \leqslant t \leqslant t_2$ and to have the constant values p_1 for $t < t_1$ and p_2 for $t > t_2$. We find that for $t \to \infty$ the field tends asymptotically to the static field corresponding to a constant dipole p_2 even in the case $p_2 \neq p_1$. This is due to the fact that the Newman–Penrose conservation laws are valid only in some region of the space–time adjacent to future null-infinity.

The formulae for the wavetail (in the region $u > t_2$) to order m/r are discussed in some detail.

1. Introduction

Consider the Maxwell theory in Minkowski space and a bounded source which is time-dependent only during a finite time interval. The retarded field produced by such a source has the form shown schematically in figure 1; the field is time-dependent only during a finite interval of retarded time, corresponding to the time-dependent state of the source.

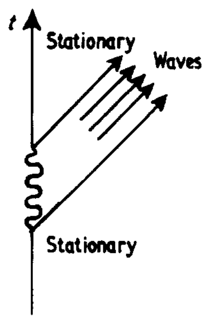

Figure 1. No tails in Minkowski space.

In general relativity the situation will be different. If we consider again a bounded source which is initially stationary and then in a state of forced vibrations during a finite time interval, the retarded field will be again stationary initially, but it will behave differently after the end of the forced vibrations. Indeed, in general relativity, we have a curved space and nonlinear field equations. Each of these features leads to a scattering of the radiation, because of which the field and its source will continue being time-dependent after the end of the forced vibrations. However, it is plausible to assume, and it has usually been assumed, that the scattered radiation will die away and the field will tend to a time-independent limit asymptotically for $t \to \infty$.

The situation appeared to be modified in a significant manner by the discovery of the Newman–Penrose conservation laws (Newman and Penrose 1965). These conservation laws are derived from the assumption that the Weyl scalars Ψ_A can be written, at large distances from the source, in the form:

$$\Psi_A = \sum_{n \leqslant 6} \frac{1}{r^n} \Psi_A^n(u, \theta, \phi) + O\left(\frac{1}{r^7}\right).$$

It is then found that five complex quantities derived from the development in spin harmonics of Ψ_0^6 are constant. In the case of a time-independent field the detailed discussion of the conserved quantities has shown that these quantities are some simple combinations of the lowest multipole moments of the system (Newman and Penrose 1968). We thus arrive at the following important restriction: an initially stationary system cannot become stationary again unless the initial and the final configuration correspond to the same values of these conserved quantities.

A similar situation presents itself in the Einstein–Maxwell theory. In this theory there are, besides the gravitational, also some electromagnetic conserved quantities, which are derived from the assumption that at large distances from the source the electromagnetic scalars Φ_A can be written in the form:

$$\Phi_A = \sum_{n \leqslant 4} \frac{1}{r^n} \Phi_A^n(u, \theta, \phi) + O\left(\frac{1}{r^5}\right), \tag{1}$$

the conserved quantities being now related to Φ_0^4. In the case of a stationary field, these quantities have been found to be some simple combinations of the lowest gravitational and electromagnetic multipoles of the system (Exton *et al* 1969). We now have the following restriction: an initially stationary Einstein–Maxwell field cannot again become stationary unless the initial and final configurations correspond to the same values of the gravitational and electromagnetic conserved quantities.

These consequences of the Newman–Penrose conservation laws are quite puzzling and several attempts have been made in order to clarify their physical meaning. As it should be expected, trying to reach an understanding of this physical meaning on the basis of the exact field equations constitutes an extremely difficult problem. We face an essentially simpler mathematical problem if we consider a test field, superimposed on a given highly symmetric background field. The simplification is due to the fact that in the test field approximation we have to deal with linear field equations; the scattering of the radiation is now due to the curvature of the space described by the background metric.

The case which will be considered in this paper is the following: a test electromagnetic field is superimposed on the Schwarzschild metric, considered as a solution of the Einstein–Maxwell field equations (vanishing background electromagnetic field).

In the case of a stationary electromagnetic test field of this kind the conserved electromagnetic quantity is found to be the product mp, m being the Schwarzschild mass and p the electric dipole moment which is the source of the test field. Since in the test field approximation the mass m is constant, it follows from the Newman–Penrose conservation law that an initially stationary test field of this kind can again become stationary only if the electric dipole moment p of the system has the same value in the final as well as in the initial configuration. It follows that in order to obtain a clarification of the situation it will be sufficient to discuss the retarded field of an electric dipole in the Schwarzschild background metric.

As we shall show in this paper, it is possible to determine the retarded field of an electric dipole which is located at the centre of symmetry and whose moment p is an arbitrarily given function of time. The conclusions, which will be obtained by the discussion of this field, are stated here in a condensed form.

(i) The quantities Φ_A can be written in the form (1) only in a certain region of the space–time adjacent to the future null-infinity determined by the condition

$$\frac{u}{r} \equiv \frac{t-r}{r} \to 0, \tag{2}$$

$u = t-r$ being the retarded time.

(ii) The field tends to a time-independent limit asymptotically for $t \to \infty$ if the dipole moment is constant initially and finally,

$$p = p_1 \qquad \text{for } t < t_1, \qquad p = p_2 \qquad \text{for } t > t_2, \tag{3}$$

even if $p_1 \neq p_2$.

It has to be stressed that the last conclusion is not related to the behaviour of the field in the neighbourhood of the null-infinity. Indeed, in order to find whether the field tends to a time-independent limit for $t \to \infty$ one has to consider the behaviour of the field at $t \to \infty$ *for any given value of the radial distance r*.

Before entering into the calculations we make the following remark. Instead of a test electromagnetic field in the Schwarzschild background metric we could consider the exact solution of the field equations of the Maxwell theory formulated in a given Riemannian space, which in this case would be the Schwarzschild space. This second formulation is conceptually simpler and in a certain sense more interesting, as it is not tied up to the test field approximation. From a practical point of view there is no difference, as we have in both cases exactly the same field equations.

2. The field equations

We shall use the Maxwell equations expressed in terms of the scalars Φ_A (Newman and Penrose 1962). In the case of the Schwarzschild background metric, which has, in radiation coordinates, the form

$$ds^2 = \left(1-\frac{2m}{r}\right) du^2 + 2du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

the vacuum Maxwell equations are:

$$\left.\begin{aligned}
\left(\partial_r+\frac{2}{r}\right)\Phi_1 &= \frac{1}{\sqrt{2}r}(\bar{\nabla}+\cot\theta)\Phi_0,\\
\left(\partial_r+\frac{1}{r}\right)\Phi_2 &= \frac{1}{\sqrt{2}r}\bar{\nabla}\Phi_1;\\
\frac{1}{\sqrt{2}r}\nabla\Phi_1 &= \left[\partial_u+\left(\frac{m}{r}-\frac{1}{2}\right)\partial_r-\frac{1}{2r}\right]\Phi_0,\\
\frac{1}{\sqrt{2}r}(\nabla+\cot\theta)\Phi_2 &= \left[\partial_u+\left(\frac{m}{r}-\frac{1}{2}\right)\partial_r+\frac{2m-r}{r^2}\right]\Phi_1;
\end{aligned}\right\}\tag{4}$$

$$\nabla = \partial_\theta+\frac{\mathrm{i}}{\sin\theta}\partial_\phi. \tag{5}$$

We can obtain an equation containing only Φ_2 by combining the second and fourth of equations (4). The result is:

$$\left(\partial_u+\frac{2m-r}{2r}\partial_r+\frac{2m-r}{r^2}\right)\partial_r(r\Phi_2)-\frac{1}{2r^2}\bar{\nabla}(\nabla+\cot\theta)(r\Phi_2) = 0. \tag{6}$$

We then arrive at a separation of the variables u, r from θ, ϕ by writing the elementary solution in the form:

$$\Phi_2 = \phi_2(u,r)\chi_2(\theta,\phi). \tag{7}$$

The final result for χ_2 is found easily to be:

$$\chi_2 = \bar{\nabla}Y_l^m \sim {}_{-1}Y_l^m, \tag{8}$$

ie χ_2 is a spin spherical harmonic of spin weight -1. For ϕ_2 we find the equation:

$$\left(\partial_u+\frac{2m-r}{2r}\partial_r\right)(r^2\,\partial_r A)+\frac{l(l+1)}{2}A = 0; \qquad A = r\phi_2. \tag{9}$$

It is to be noted that, when we have determined a solution Φ_2 of (6) for any $l \neq 0$, we can determine the corresponding Φ_1 and Φ_0 respectively from the second and the first of equations (4) without any new integration. Indeed, if we write:

$$\Phi_1 = \phi_1(u,r)\chi_1(\theta,\phi), \qquad \Phi_0 = \phi_0(u,r)\chi_0(\theta,\phi), \tag{10}$$

we find (without normalization):

$$\chi_1 = Y_l^m, \qquad \chi_0 = \frac{-1}{l(l+1)}\nabla Y_l^m; \tag{11}$$

$$\phi_1 = \sqrt{2}\,\partial_r(r\phi_2), \qquad \phi_0 = \frac{\sqrt{2}}{r}\partial_r(r^2\phi_1). \tag{12}$$

It is therefore sufficient to determine ϕ_2 by integrating equation (9).

The case of the dipole, in which we are interested here, corresponds to $l = 1$. Equation (9) takes then the form:

$$\left[\partial_u+\left(\frac{m}{r}-\frac{1}{2}\right)\partial_r\right](r^2\partial_r A)+A = 0. \tag{13}$$

More exactly we have to determine the retarded field of an electric dipole p which satisfies the relations (3), depending arbitrarily on t in the interval $t_1 < t < t_2$. We shall simplify the problem by assuming $p_1 = 0$: because of the linearity of the Maxwell equations in the test field approximation we simply have in the case $p_1 \neq 0$ an additional static field which is unimportant for the questions in which we are interested here. The retarded character of the field produced by the dipole which has $p_1 = 0$ will be assured if the field vanishes in the region defined by the relation $u < t_1$.

We shall assume in this paper that t_1 is finite. The case $t_1 \to -\infty$, which has been discussed by Press and Bardeen (1971), will not be considered here.

Again, because of the linearity of the field equations it will be useful to determine the retarded field of an electric dipole, the moment of which is a step function:

$$\begin{aligned} &p(t) = S(t-t_0); \\ &S(x) = 0 \qquad \text{for } x < 0, \qquad S(x) = 1 \qquad \text{for } x \geqslant 0. \end{aligned} \tag{14}$$

This is the problem which we shall discuss first. The field of a dipole, whose moment is a smooth function of t (with $p_1 = 0$) will be determined in § 8 by superposition of solutions corresponding to dipoles of the form (14). Note that the infinite electromagnetic energy emitted by a dipole of the form (14) is eliminated by the superposition of such dipoles leading to a smooth function $p(t)$.

If we put $m = 0$ in (9) or (13), we obtain the field equation for Minkowski space. The retarded solution of (13) is in this case

$$A \equiv A_0 = \ddot{p}(u)+\frac{1}{r}\dot{p}(u)+\frac{1}{2r^2}p(u); \qquad \dot{p} \equiv \frac{\mathrm{d}p}{\mathrm{d}u}. \tag{15}$$

The function $p(u)$ represents the dipole moment as a function of the retarded time u. We shall see that the expression A_0 is the leading term of the solution of equation (13) when $m \neq 0$.

3. The static field

For a constant dipole we obtain from (15) the static field in the case $m = 0$:

$$A_{0\mathrm{st}} = \frac{1}{2r^2}p, \qquad p = \text{constant}. \tag{15'}$$

We now consider the case $m \neq 0$ and we determine the static solution of equation (13), describing the field of a constant dipole p. This solution is obtained immediately if we write it in the form of a power series in $1/r$. The final result is:

$$A_{\mathrm{st}} = \frac{p}{2r^2}+\frac{mp}{2r^3}\left[1+\frac{3\,.\,4}{4\,.\,5}\frac{2m}{r}+\frac{3\,.\,4}{5\,.\,6}\left(\frac{2m}{r}\right)^2+\ldots\right]. \tag{16}$$

The series appearing in the bracket of (16) converges absolutely for any $r > 2m$.

The solution (16) can also be written in a closed, but rather clumsy, form:

$$A_{st} = \frac{p}{2r^2} + \frac{3p}{4m^2} f(\alpha), \qquad \alpha = \frac{2m}{r}; \tag{17}$$

$$f(\alpha) = -\frac{\alpha^2}{6} - \frac{\alpha}{2} + 1 + \frac{1-\alpha}{\alpha} \lg(1-\alpha). \tag{17'}$$

4. The time-dependent solution of equation (13) to order m/r

The retarded field of a time-dependent dipole $p(t)$ can be written, in the case $m \neq 0$, in the form:

$$A = A_0 + a, \tag{18}$$

A_0 being given by (15). Introducing the expression (18) into equation (13) we find the following equation for a:

$$(\partial_u - \tfrac{1}{2}\partial_r)(r^2 \partial_r a) + a + \frac{mp}{r^3} + \frac{m}{r} \partial_r(r^2 \partial_r a) = 0. \tag{19}$$

Equation (19) shows that at distances $r \gg 2m$ the quantity a will be small, of the order m/r. It is instructive to discuss first the equation resulting from (19) by linearization with respect to m/r. This linearization is obtained at once if we omit the last term in (19). The linearized equation is therefore:

$$(\partial_u - \tfrac{1}{2}\partial_r)(r^2 \partial_r a) + a + \frac{mp}{r^3} = 0. \tag{20}$$

The solution of this equation will represent the main part of the scattering of the outgoing dipole wave on the curvature of the Schwarzschild space.

The solution of equation (20) can be found if we write it in the form:

$$a = \sum_{n \geqslant 4} \frac{a_n}{r^n}, \tag{21}$$

the coefficients $a_4, a_5, \ldots$ being functions of u only. Introducing the expression (21) in (20) we find the relations:

$$\dot{a}_4 = \frac{mp}{4}; \qquad \dot{a}_n = -\frac{n-3}{2} a_{n-1} \qquad \text{for } n > 4. \tag{22}$$

For the dipole moment (14),

$$p(u) = S(u - u_0), \tag{23}$$

the final result is:

$$a = \frac{m}{2r^3} \left[\frac{u-u_0}{2r} - \left(\frac{u-u_0}{2r}\right)^2 + \left(\frac{u-u_0}{2r}\right)^3 - \ldots \right] S(u-u_0). \tag{24}$$

The series appearing in the right-hand side of equation (24) converges absolutely if

$$x \equiv \frac{u-u_0}{2r} < 1 \tag{25}$$

and therefore the expression (24) represents the solution of equation (20) in the region defined by (25). The condition (25) is certainly satisfied in the neighbourhood of null-infinity. However, in order to see whether the field tends to a time-independent limit at $t \to \infty$ we need to consider also the case $x \geqslant 1$, in which the series (24) cannot be used. This difficulty can be overcome easily because the series (24) is of a very simple type and we can write at once its sum in a closed form:

$$x - x^2 + x^3 - \ldots = \frac{x}{1+x}. \tag{26}$$

It follows that we shall have:

$$a = \frac{m}{2r^3} \frac{x}{1+x} S(x), \tag{27}$$

this form of the solution of equation (20) being valid for all values of x.

When $u \to \infty$, we have for any given r, $x \to \infty$, and consequently we find from (27):

$$\lim_{u \to \infty} A = \frac{1}{2r^2} + \frac{m}{2r^3}. \tag{28}$$

Comparing this limit with (16) we see that it is identical, in the approximation considered here, with the static field of a dipole having the constant moment $p = 1$.

5. The general form of the solution of equation (19)

The general structure of the solution of equation (19) will be found if we assume that this solution can also be written in the form (21). Introducing (21) in equation (19) we find now the following recurrence formulae:

$$\begin{aligned} &\dot{a}_4 = \frac{mp}{4}, \qquad \dot{a}_5 = -a_4; \\ &\dot{a}_n = -\frac{n-3}{2} a_{n-1} + \frac{(n-2)(n-3)}{n} m a_{n-2} \qquad \text{for } n > 5. \end{aligned} \tag{29}$$

In the case of the dipole (23), which interests us here, we find by integration of (29):

$$\begin{aligned} &a_4 = \frac{m}{2} \frac{u-u_0}{2} S(u-u_0); \qquad a_5 = -\frac{m}{2}\left(\frac{u-u_0}{2}\right)^2 S(u-u_0); \\ &a_6 = \left[\frac{m}{2}\left(\frac{u-u_0}{2}\right)^3 + m^2\left(\frac{u-u_0}{2}\right)^2\right] S(u-u_0); \ldots . \end{aligned} \tag{30}$$

We see that the general term a_n will be a polynomial in $(u-u_0)/2$:

$$a_n = \left[\alpha_n m \left(\frac{u-u_0}{2}\right)^{n-3} + \beta_n m^2 \left(\frac{u-u_0}{2}\right)^{n-4} + \ldots\right] S(u-u_0), \tag{30'}$$

$\alpha_n, \beta_n, \ldots$ being numerical coefficients. Introducing the values (30) or (30′) in (21) we find the solution of equation (19) in the form of a double series, the two variables being x and m/r. The series does certainly converge for sufficiently small values of the variables.

Grouping together the terms containing the same power of m/r we obtain the form:

$$a = \frac{1}{r^2}\left(\frac{m}{r}\tilde{f}_1(x) + \left(\frac{m}{r}\right)^2 \tilde{f}_2(x) + \ldots\right) S(u-u_0). \tag{31}$$

The coefficients $\tilde{f}_1, \tilde{f}_2, \ldots$ are power series in x of the form:

$$\tilde{f}_1 = \sum_{n\geqslant 4} \alpha_n x^{n-3}, \qquad \tilde{f}_2 = \sum_{n\geqslant 6} \beta_n x^{n-4}, \ldots. \tag{32}$$

In the preceding section we have determined the series $\tilde{f}_1$ and its sum. The series $\tilde{f}_2, \tilde{f}_3, \ldots$ are essentially more complicated and determining their sum in a similar way should be difficult. It will be easier to determine them as solutions of certain differential equations which will be established now.

The basic idea consists in considering A formally as a power series in m:

$$A = A_0 + mA_1 + m^2 A_2 + \ldots. \tag{33}$$

Comparing (33) with (31) we find at once:

$$A_n = \frac{f_n}{r^{n+2}}, \qquad f_n = \tilde{f}_n S(x) \qquad \text{for } n \geqslant 1. \tag{34}$$

Introducing (33) in (13) we find:

$$(\partial_u - \tfrac{1}{2}\partial_r)(r^2\partial_r A_0) + A_0 = 0; \tag{35a}$$

$$(\partial_u - \tfrac{1}{2}\partial_r)(r^2\partial_r A_1) + A_1 + \frac{1}{r}\partial_r(r^2\partial_r A_0) = 0; \tag{35b}$$

$$(\partial_u - \tfrac{1}{2}\partial_r)(r^2\partial_r A_n) + A_n + \frac{1}{r}\partial_r(r^2\partial_r A_{n-1}) = 0 \qquad \text{for } n > 1. \tag{35c}$$

The solution A_0 of (35a) is given by (15). From this value of A_0 we calculate the last term of equation (35b). The result is:

$$\frac{1}{r}\partial_r(r^2\partial_r A_0) = \frac{1}{r^3}p(u).$$

We then introduce in (35b) the value of A_1, given by (34). Taking into account the relations:

$$\frac{\partial x}{\partial u} = \frac{1}{2r}, \qquad \frac{\partial x}{\partial r} = -\frac{x}{r} \tag{36}$$

we find finally, in the case of the dipole moment (23), the equation:

$$(x+x^2)f_1'' + (4+6x)f_1' + 4f_1 = 2S(x), \qquad f' \equiv \frac{\mathrm{d}f}{\mathrm{d}x}. \tag{37}$$

Similarly we find from (35c), when we introduce the expression (34) for A_n:

$$(x+x^2)f_n'' + \{(n+3) + 2(n+2)x\}f_n' + (n+3)nf_n$$
$$= 2x^2 f_{n-1}'' + 4(n+1)xf_{n-1}' + 2n(n+1)f_{n-1} \qquad \text{for } n > 1. \tag{38}$$

6. The solution of equation (19) to order $(m/r)^2$

In order to obtain the retarded field of the dipole (23) we have to determine the solutions of equations (37) and (38) which vanish for $x < 0$. We verify directly that the solution of equation (37) is the coefficient of m/r^3 in (27):

$$f_1 = \frac{x}{2(1+x)} S(x). \tag{39}$$

The solution of equation (38) can be written in a simple integral form. We shall determine this form here only for the case $n = 2$.

The right-hand side of equation (38) for $n = 2$ is:

$$\alpha_1(x) = 2x^2 f''_1 + 12xf'_1 + 12f_1.$$

Taking into account the relation

$$\{xS(x)\}' = S(x),$$

which leads to:

$$xS' = x^2 S'' = 0,$$

we find:

$$\alpha_1(x) = 2xS(x)\left(\frac{3}{1+x} + \frac{2}{(1+x)^2} + \frac{1}{(1+x)^3}\right). \tag{40}$$

Multiplying equation (38) corresponding to $n = 2$ by x^4 we find:

$$\{(x^5+x^6)f'_2 + 2x^5 f_2\}' = x^4\alpha_1(x).$$

The solution of this equation which vanishes for $x < 0$ is:

$$(x^5+x^6)f'_2 + 2x^5 f_2 = \int_{-\infty}^{x} x^4\alpha_1(x)\,\mathrm{d}x.$$

The second integration is obtained if we multiply this equation by $(1+x)/x^5$. The final result is

$$f_2 = \frac{S(x)}{(1+x)^2}\int_0^x \frac{1+x}{x^5}\left(\int_0^x x^4\alpha_1(x)\,\mathrm{d}x\right)\mathrm{d}x. \tag{41}$$

The detailed expression for f_2 is rather long. We give it without calculations:

$$f_2 = \{B(x)\lg(1+x) + C(x)\}S(x); \tag{42}$$

$$\left.\begin{aligned} B(x) &= \frac{3}{2x^4} - \frac{1}{x^3} + \frac{1}{2x^2} - \frac{1}{(1+x)^2}, \\ C(x) &= -\frac{3}{2x^3} + \frac{7}{4x^2} - \frac{3}{2x} - \frac{77}{120(1+x)^2} + \frac{1}{1+x} + \frac{3}{5}. \end{aligned}\right\} \tag{42'}$$

It is not difficult to show that for $x \ll 1$ one can write f_2 as the product of $S(x)$ by an infinite polynomial in x starting with the term x^2, in agreement with the formula (32).

For $x \to \infty$ we find:

$$\lim_{x \to \infty} \tilde{f}_2 = \tfrac{3}{5}. \tag{43}$$

The solution ϕ_2 of equation (9) corresponding to the dipole (23) is derived from equations (33) and (34). We find:

$$\phi_2 = \frac{1}{r}\ddot{S}(u-u_0)+\frac{1}{r^2}\dot{S}(u-u_0)+\left(\frac{1}{2r^3}+\frac{m}{r^4}\tilde{f}_1+\frac{m^2}{r^5}\tilde{f}_2+\ \ldots\right)S(u-u_0). \tag{44}$$

We have already determined the functions $\tilde{f}_1(x)$ and $\tilde{f}_2(x)$.

The following conclusion can be drawn from the expression (44) for ϕ_2. When $u/r \to \infty$, ie $x \to \infty$, the field ϕ_2 tends, in the approximation considered here, to a time-independent limit:

$$\lim_{x \to \infty} \phi_2 = \frac{1}{2r^3}+\frac{m}{2r^4}+\frac{3m^2}{5r^5}+O\left(\frac{m^3}{r^3}\right). \tag{45}$$

Comparing with (16) we see that this limit is identical, to order $(m/r)^2$, with the quantity $\phi_{2\text{st}}$ which describes the static field of the constant dipole $p = 1$. A similar result holds for the quantities ϕ_1 and ϕ_0, as one can prove with the help of equations (12).

The mathematical method used in §§ 5 and 6 is similar to that used by Bardeen and Press (1973). The fact that the field tends asymptotically to a time-independent limit for $t \to \infty$ has been proved to order m/r by Press and Bardeen (1971). The general proof to all orders will be given in a forthcoming paper by Linet (1975).

7. The Newman–Penrose conservation laws

In order to compare the result (44) with the form (1) assumed by Newman and Penrose, we see at once that we need to consider the behaviour of the function $\tilde{f}_1$ only. According to (24) this function can be written as a converging power series of x for $x < 1$. It follows that in the region defined by the relation $x < 1$ the function ϕ_2 is of the form (1). On the contrary, for $x \geqslant 1$ the development of $\tilde{f}_1$ in power series of x is not possible. Consequently the solution (44) is not of the form (1) in the region $x \geqslant 1$. We conclude that the Newman–Penrose conservation laws are valid only in the region corresponding to $x < 1$.

The relation $x = 1$ is equivalent, because of the definitions (25) and (2) of x and u, to:

$$ct-u_0 = 3r. \tag{46}$$

(We have re-introduced in this relation the missing factor c.) In the (t, r) diagram this is the equation of a straight line describing the motion of a particle starting from $r = 0$ at $t = u_0/c$ and moving with the velocity $c/3$. This straight line divides the plane (t, r) into the regions I and II (figure 2). The corresponding Penrose diagram is given in Press and Bardeen (1971).

The Newman–Penrose conservation laws are valid in region I but not in II. The points of the future null-infinity, $u/r \to 0$ for any given value of u, are all contained in region I. But the points with $u/r \to \infty$ for any given r are contained in region II. This explains the fact that the field tends to a time-independent limit in spite of the restrictions imposed by the Newman–Penrose conservation laws.

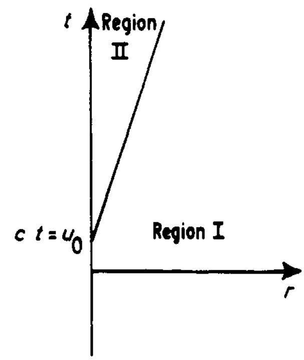

Figure 2. The regions I and II.

8. The field of a smooth function $p(t)$

We now consider a dipole moment represented by the smooth function $p(t)$ of the form shown on figure 3. As we remarked in § 2, it is sufficient to consider the case with $p_1 = 0$, since for $p_1 \neq 0$ we have only to add to the solution a static field corresponding to the constant dipole p_1. We shall determine the retarded field produced by the dipole $p(t)$, restricting ourselves to the terms of order m/r.

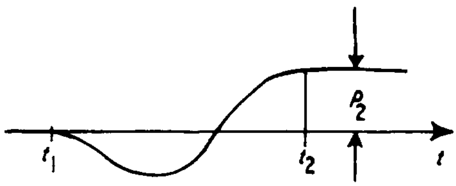

Figure 3. A smooth function $p(t)$.

We start with the remark that the dipole $p(t)$ can be obtained by the superposition of dipoles of the form (14), according to the relation:

$$p(t) = \int_{t_1}^{t_2} f(t_0)S(t-t_0)\,\mathrm{d}t_0; \qquad f(t) = \frac{\mathrm{d}p(t)}{\mathrm{d}t}. \tag{47}$$

We have already determined the retarded field produced by the dipole $S(t-t_0)$: it is the field given by equation (44), which we shall denote in this section by $\phi_{2(u_0)}$. Because of the linearity of the field equation the field corresponding to the dipole $p(t)$ given by (47) will be:

$$\phi_2(u, r) = \int_{u_1}^{u_2} f(u_0)\phi_{2(u_0)}\,\mathrm{d}u_0. \tag{48}$$

It follows from this formula and from the result obtained at the end of § 6 for $\phi_{2(u_0)}$ that the retarded field of the dipole $p(t)$ tends, for any given r and $t \to \infty$, to the static field of the constant dipole p_2.

After some partial integrations based on the second of equations (47) we find from (48):

$$\phi_2(u,r) = \frac{1}{r}\ddot{p}(u)+\frac{1}{r^2}\dot{p}(u)+\frac{1}{2r^3}p(u)+\frac{m}{2r^4}\int_{u_1}^{u_2} f(u_0)\left(1-\frac{1}{1+x}\right)S(u-u_0)\,\mathrm{d}u_0+\mathrm{O}\left(\frac{m^2}{r^2}\right). \quad (49)$$

It is to be noted that the form (1) can now be obtained only in the region of the plane (t, r) which is on the right of the line $c(t-t_1) = 3r$.

We mention that the retarded field produced by the dipole $p(t)$ can be determined also with the help of the equation

$$p(t) = \int p(t_0)\delta(t-t_0)\,\mathrm{d}t_0,$$

$\delta(t-t_0)$ being the Dirac function:

$$\delta(t-t_0) = \frac{\mathrm{d}}{\mathrm{d}t}S(t-t_0).$$

One needs in this case the field produced by the dipole $\delta(t-t_0)$. One can show easily that this field is given by the derivative of $-\phi_{2(u_0)}$ with respect to u_0. For a function $p(t)$ of the form shown on figure 3 it is more convenient to use the representation (47).

9. Formulae for the wavetail to order *m*/*r*

Finally we shall derive and discuss the formula giving the field in the region defined by

$$u > u_2,$$

in which there is only scattered radiation (wavetail). In this case the first two terms in (49) vanish. The next two terms represent the asymptotic static field, for $t \to \infty$, to the order m/r. We can therefore write:

$$\phi_2 = \phi_{2\mathrm{st}}-\frac{m}{2r^4}\int_{u_1}^{u_2}\frac{1}{1+x}f(u_0)\,\mathrm{d}u_0+\mathrm{O}\left(\frac{m^2}{r^2}\right). \quad (50)$$

The second term on the right-hand side of (50) describes the wavetail to order m/r.

From the definition of x in (25) we have:

$$\frac{1}{1+x} = \frac{2r}{2r+u-u_0} = \frac{2r}{t+r-u_0}.$$

Consequently we can write (50) in the following form:

$$\phi_2 = \phi_{2\mathrm{st}}-\frac{m}{r^3}I(\lambda)+\mathrm{O}\left(\frac{m^2}{r^2}\right); \quad (51)$$

$$I(\lambda) = \int_{u_1}^{u_2} f(u_0)\frac{\mathrm{d}u_0}{\lambda-u_0}, \qquad \lambda = t+r. \quad (52)$$

In the approximation considered here λ is the advanced time and λ = constant is the equation of a past light cone.

Using the second of equations (47) we can transform (52) into the form:

$$I(\lambda) = \frac{p_2}{\lambda - u_2} - \int_{u_1}^{u_2} p(u_0) \frac{\mathrm{d}u_0}{(\lambda - u_0)^2}. \tag{52'}$$

The integral (52) depends on λ only and consequently it has the same value at all points of the (past) light cone $t+r$ = constant. For an oscillating dipole moment $p(u)$ the function $I(\lambda)$ will vanish for certain values of λ. In order to determine these values one has to know the exact form of the function $p(u)$. Note that the field functions ϕ_1 and ϕ_0, which are to be determined from equation (12), will in general not vanish for the same values of λ.

We shall derive the asymptotic form of the integral (52) for values of λ satisfying the condition:

$$\lambda \gg |u_1| \qquad \text{and} \qquad |u_2|. \tag{53}$$

It will be convenient to choose the point $t = 0$ on the time axis so as to have

$$u_2 = 0. \tag{54}$$

The assumption (53) reduces then to

$$\lambda \gg |u_1| = -u_1. \tag{53'}$$

In this case the values of u_0 considered in the integral (52) satisfy also the condition:

$$\lambda \gg |u_0|.$$

Consequently we can write:

$$\frac{1}{(\lambda - u_0)^2} = \frac{1}{\lambda^2}\left(1 - \frac{u_0}{\lambda}\right)^{-2} = \frac{1}{\lambda^2}\left(1 + 2\frac{u_0}{\lambda} + 3\frac{u_0^2}{\lambda^2} + \dots\right). \tag{55}$$

Introducing the development (55) in (52′) we find, remembering that $f(u_0) = \mathrm{d}p(u_0)/\mathrm{d}u_0$, $p_1 = 0$ and $u_2 = 0$:

$$I(\lambda) = \frac{1}{\lambda} p_2 - \frac{1}{\lambda^2} \int_{u_1}^{u_2} p(u_0)\, \mathrm{d}u_0 - \frac{2}{\lambda^2} \int_{u_1}^{u_2} p(u_0) u_0 \, \mathrm{d}u_0 + \dots . \tag{56}$$

Equation (51) takes then the form:

$$\phi_2(u, r) = \phi_{2\mathrm{st}} - \frac{m}{r^3}\left(\frac{p_2}{t+r} - \frac{1}{(t+r)^2} \int_{u_1}^{u_2} p(u_0)\, \mathrm{d}u_0 - \frac{2}{(t+r)^3} \int_{u_1}^{u_2} p(u_0) u_0 \, \mathrm{d}u_0 - \dots\right). \tag{57}$$

This formula determines the detailed manner in which the field tends to its time-independent limit for $t \to \infty$.

Equation (57) leads to the following conclusions. If $p_2 \neq 0$, ie if the final dipole moment differs from the initial one, then for any given value of r the difference $\phi_2 - \phi_{2\mathrm{st}}$ tends to zero for $t \to \infty$ as $1/t+r$. If $p_2 = 0$ but $\int p(u_0)\, \mathrm{d}u_0 \neq 0$, then this difference tends to zero as $1/(t+r)^2$. If we have:

$$p_2 = 0, \qquad \int_{u_1}^{u_2} p(u_0)\, \mathrm{d}u_0 = 0; \qquad \int_{u_1}^{u_2} p(u_0) u_0 \, \mathrm{d}u_0 \neq 0,$$

then this difference tends to zero as $(t+r)^{-3}$ and so on: imposing on $p(u_0)$ the μ conditions ($\mu \geqslant 3$):

$$p_2 = 0, \qquad \int p(u_0)\,\mathrm{d}u_0 = 0, \qquad \int p(u_0)u_0\,\mathrm{d}u_0 = 0, \qquad \ldots$$

we can obtain that this difference tends to zero as $(t+r)^{-(\mu+1)}$.

In the neighbourhood of the future null-infinity, where we have $x \ll 1$, we can obtain a development of the integral appearing in equation (50) based on the relation:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots.$$

The difference $\phi_2 - \phi_{2st}$ will then appear as an infinite polynomial in u. This is a special case of the general result obtained for non-radiative fields (Moret-Bailly and Papapetrou 1967, Moret-Bailly 1968). The formula (57) has the advantage of being valid in the whole shaded region of figure 4 and not only near null-infinity.

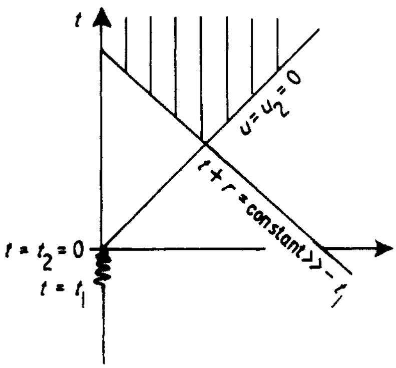

Figure 4. The light cones $u = 0$ and $t+r$ = constant.

Electromagnetic wavetails have been considered to order m/r by Rotenberg (1971), using a different method (double series expansion). Gravitational wavetails have been discussed, again to order m/r and using the method of double series expansion, by Bonnor and Rotenberg (1966) and by Hunter and Rotenberg (1969). A common feature of the results obtained in all these papers is that the scattered radiation depends on the advanced time $t+r$.

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